Relating Subgroups in Galois Cohomology
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Assumptions: Let G be a group with subgroup H.

Question: How can we relate $H^i(G, A)$ to $H^i(H, A)$ for any G-module A?

The above question makes sense because any G-action on A also induces an H-action so we can talk about $H^i(H, A)$. We will use the fact that every G-module is naturally an H-module tacitly throughout this talk.

In particular $\mathbb{Z}[G]$ has a natural G-module structure, so it makes sense to define the following H-module:

Definition. *Given an H-module A, we define*

$$M_H^G(A) = \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

M_H^G is certainly an H-module, but it is also a G-module. For $\sigma \in G$, we define a G-action on M_H^G by $\sigma(\phi(g)) = \phi(g\sigma)$ where $g \in \mathbb{Z}[G]$. To clear up notation, $g\sigma$ is considered as an element of $\mathbb{Z}[G]$.

With this definition in place, we can present our first result that suggests there is a connection between $H^i(G, A)$ and $H^i(H, A)$.

Lemma. *Let A be an H-module and M a G-module, then*

$$\mathrm{Hom}_G(M, M_H^G(A)) \cong \mathrm{Hom}_H(M, A).$$

Proof. Given an H-morphism $\lambda : M \to H$, define the G-morphism by

$$m \mapsto (g \mapsto \lambda(gm))$$

where, to clear up notation, gm means g acting on m.

In the other direction, given a G-morphism $m \mapsto \phi_m$, we define an H-morphism by $m \mapsto \phi_m(1)$. □

Unsurprisingly, when we pass to cohomology we get an isomorphism.

Corollary (Shapiro's Lemma)**.** *Given an H-module A, we have the following isomorphisms for all $i \geq 0$:*

$$H^i(G, M_H^G(A)) \cong H^i(H, A).$$

In the case of $H = \{1\}$ (i.e., in the category of abelian groups), we obtain an immediate corollary:

Corollary. $H^i(G, M_{\{1\}}^G(A)) = \{1\}$ *for all $i > 0$.*

Proof (Corollary). A projective H-resolution of $\mathbb{Z}$ is $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ (because $\mathbb{Z}[H] = \mathbb{Z}[\{1\}] = \mathbb{Z}$). Thus, $H^i(H, A) = 0$ for $i > 0$, so the result follows □

We have kind of answered our original question, i.e. we can interpret group cohomology over H and group cohomology over G (with a slightly different module). But, we really wanted to compare cohomology for the same G-module A. To get to a place where we can compare cohomology in such a way, we need to spend a little bit more time in the $H = \{1\}$ case.

Definition. *If $H = \{1\}$, we write $M_H^G(A) = M^G(A)$ and call it the co-induced module associated with A.*

Here are some basic facts about the co-induced module:

- There is a natural injection $A \hookrightarrow M^G(A)$ given by $a \mapsto (g \mapsto ga)$;
- If G is finite and A is an abelian group (using that $\{1\}$-module is an abelian group), a choice of $\mathbb{Z}$ basis for $\mathbb{Z}[G]$ induces an isomorphism $M^G(A) = A \otimes_{\mathbb{Z}} M^G(A)$ as abelian groups.

We will come back to the co-induced module later, but we now will introduce some maps on cohomology.

Definition. *Let A be a G-module. There is a G-morphism $A \to M_H^G(A)$ given by first considering the isomorphism $A \cong \mathrm{Hom}_G(\mathbb{Z}[G], A)$ then, second, considering the associated G-morphism as an H-morphism. Thus, we have constructed a G-morphism $A \to \mathrm{Hom}_H(\mathbb{Z}[G], A) = M_H^G(A)$. If we take cohomology $H^i(G, \cdot)$ we get induced abelian group morphisms $H^i(G, A) \to H^i(M_H^G(A))$ for all i. It follows by Shapiro's Lemma that we get an abelian group morphisms*

$$\mathrm{res} : H^i(G, A) \to H^i(H, A)$$

for all i which we call the restriction map.

In the case of $i = 0$, res is just the usual inclusion $A^G \to A^H$.

Now, if $[G : H]$ is finite, we get an additional maps in the opposite direction of the restriction maps

Definition. *Assume that $[G : H]$ is finite and let A be a G-module. For $\phi \in \mathrm{Hom}_H(\mathbb{Z}[G], A)$ we define a morphism $\phi_H^G \in \mathrm{Hom}_H(\mathbb{Z}[G], A) \cong A$ as follows:*

Let $\rho_1, \ldots, \rho_n$ be a system of left coset representatives for H in G. Then we define ϕ_H^G via

$$\phi_H^G(x) = \sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x).$$

($\rho_j^{-1} x$ means ρ_j^{-1} is acting on x). Since ϕ is a H-morphism, ϕ_H^G does not depend on choice of coset representative.

So far, ϕ_H^G is, a priori, only an H-morphism. To see ϕ_H^G is a G-morphism, let $\sigma \in G$ and notice that $\sigma^{-1}\rho_j$ forms another system of left coset representatives. Therefore, we find that

$$\sigma \left(\sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x) \right) = \sigma \left(\sum_{j=1}^{n} \sigma^{-1} \rho_j \phi((\sigma^{-1}\rho_j)^{-1} x) \right) = \sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} \sigma x)$$

which tells us that $\phi_H^G \in \mathrm{Hom}_G(\mathbb{Z}[G], A)$.

In short, $\phi \mapsto \phi_H^G$ gives us a map $\mathrm{Hom}_H(\mathbb{Z}[G], A) \to \mathrm{Hom}_G(\mathbb{Z}[G], A) \cong A$. Using the same process as above (taking cohomology then applying Shapiro's Lemma), we get maps on cohomology:

$$\mathrm{cor} : H^i(H, A) \to H^i(G, A)$$

for all i. We call these abelian group morphisms the corestriction maps.

In the case of $i = 0$, cor is the morphism $A^H \to A^G$ given by $x \mapsto \sum \rho_j x$ where ρ_j are the coset representatives of H in G.

Proposition. *Let $[G : H] = n$ be finite. Then* $\text{cor} \circ \text{res} : H^i(G, A) \to H^i(G, A)$ *is given by multiplication by n for all i.*

Proof. Let $\phi \in \text{Hom}_G(\mathbb{Z}[G], A)$ and $x \in \mathbb{Z}[G]$. The image corresponding to

$$\text{cor} \circ \text{res} : \text{Hom}_G(\mathbb{Z}[G], A) \to \text{Hom}_G(\mathbb{Z}[G], A)$$

is given by

$$\phi_H^G(x) = \sum \rho_j \phi(\rho_j^{-1} x) = \sum \rho_j \rho_j^{-1} \phi(x) = n\phi(x)$$

(because ϕ is a G-morphism - not just an H-morphism). It follows that the morpism induced on cohomology is also multiplication by n. □

Corollary. *Let G be a finite group of order n. Then for all $i > 0$, every element of $H^i(G, A)$ has order dividing n.*

Proof. Take $H = \{1\}$ in the above proposition. □

Definition. *Assume that H is a normal subgroup of G, and let A be a G-module. Notice that A^H is stable under action of G (i.e. for $\sigma \in G$ and $a \in A^H$, $\sigma a \in A^H$). Hence, A^H has an induced structure of a G/H-module.*

Now, take a projective resolution $P_\bullet$ of $\mathbb{Z}$ as a trivial G-module and a projective resolution $Q_\bullet$ of $\mathbb{Z}$ as a trivial G/H-module. Each Q_i has the structure of a G-module via the projection $GtoG/H$ so by the Horseshoe Lemma, we get a G-module complex morphism $P_\bullet \to Q_\bullet$. It follows that we also get G-module complex morphisms $\text{Hom}_G(Q_\bullet, A^H) \to \text{Hom}_G(P_\bullet, A^H)$*. Notice that* $\text{Hom}_G(Q_\bullet, A^H) = \text{Hom}_{G/H}(Q_\bullet, A^H)$ *by the comment above, so we actually have abelian group morphism* $\text{Hom}_{G/H}(Q_\bullet, A^H) \to \text{Hom}_G(P_\bullet, A^H)$*. By taking cohomology, we get maps*

$$H^i(G/H, A^H) \to H^i(G, A^H)$$

for all i. Using usual cohomology arguments, these morphisms do not depend on the choice of projective resolution.

On the other hand, we have a natural G-module morphism $A^H \to A$ which gives us an abelian group morphism on cohomology: $H^i(G, A^H) \to H^i(G, A)$.

We define the inflation maps to be the composition

$$\text{inf} : H^i(G/H, A^H) \to H^i(G, A^H) \to H^i(G, A)$$

for all i.

Lemma. *Let A be a G-module and H a normal subgroup of G. Then*

$$M^G(A)^H \cong M^{G/H}(A)$$

and

$$H^i(H, M^G(A)) = 0$$

for all $i > 0$.

Proof. The first isomorphism comes from the following isomorphism of abelian groups

$$\mathrm{Hom}(\mathbb{Z}[G], A)^H \cong \mathrm{Hom}(\mathbb{Z}[G/H], A).$$

With regards to the second isomorphism, notice that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module so $M^G(A) \cong \oplus M^H(A)$. However, using a general fact about cohomology,

$$H^i(H, M^G(A)) = H^i(H, \oplus M^H(A)) \cong \oplus H^i(H, M^H(A))$$

We notices earlier that $H^i(H, M^H(A)) = \{1\}$, so the result follows. ☐

Theorem. *Let A be a G-module and H a normal subgroup of G. Then we have an exact sequence:*

$$0 \to H^1(G/H, A^H) \xrightarrow{\inf} H^1(G, A) \xrightarrow{\mathrm{res}} H^1(H, A)^{G/H} \xrightarrow{\tau}$$
$$\to H^2(G/H, A^H) \xrightarrow{\inf} H^2(G, A).$$

We call τ the transgression map.

Proof. omitted. ☐

Corollary. *In the situation above, let $i > 1$ and assume that $H^j(H, A)$ are trivial for $1 \leq j \leq i-1$. Then there is an exact sequence:*

$$0 \to H^i(G/H, A^H) \xrightarrow{\inf} H^i(G, A) \xrightarrow{\mathrm{res}} H^i(H, A)^{G/H} \xrightarrow{\tau_{i,A}}$$
$$\to H^{i+1}(G/H, A^H) \xrightarrow{\inf} H^{i+1}(G, A).$$

We also call $\tau_{i,A}$ transgression maps.

Proof. Dimension shift.

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REPRESENTATIVES OF TRANSGRESSION MAP AND INFLATION MAP IN TERMS OF COCYCLES

As an interesting aside, we can describe inflations in terms of group extensions.

Example: Given a group G and G-module A, we shall consider group extensions $1 \to A \to E \to G \to 0$ where E is a group (not necessarily abelian). There is a bijection between 2-cocycles in $Z^2(G, A)$ and extensions of the above form (upto usual isomorphism of short exact sequences). You can read about this in Weibel Section 6.6. Given an extension $1 \to A \to E \to G \to 0$, we set $c(E)$ to be the image of the 2-cocycle in $H^2(G, A)$. (An interesting fact is that $c(E) = 0$ if, and only if, E is the semidirect product of A and G).

Given a morphism $\phi : A \to B$ of G-modules, the natural map on cohomology $\phi_* : H^2(G, A) \to H^2(G, B)$ induces the following on extensions. If $1 \to A \xrightarrow{i} E \to G \to 0$ then we get an extension $1 \to B \to F \to G \to 0$ where $F = B \times E/(\phi(a), (i(a))^{-1})_{a \in A}$ We then get $\phi_*(c(E)) = c(\phi_*(E))$.

For the inflation map, we get something similar, so let $0 \to A \to E \xrightarrow{\pi} G/H \to 1$ be an extension corresponding to $c(E) \in H^2(G/H, A)$. Then, $\inf(c(E)) = c(\rho^*(E))$ where ρ^* is defined as $E \times G/(e, g)$ that satisfy $\pi(e) = \rho(g)$.